The elliptic billiard: subtleties of separability

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Abstract. Some of the subtleties of the integrability of the elliptic quantum billiard are discussed. Considering a well known classical constant of the motion in the quantum case, we find that a naive calculation of the commutator with the Hamiltonian does not show whether or not it is zero. It is shown how this problem can be solved. A geometric picture is given that reveals why levels of a separable system cross. It is shown that the repulsions found by Ayant and Arvieu are computational effects and that the method used by Traiber *et al* is related to the present picture which explains the crossings they find. An asymptotic formula for the energy levels is derived and it is found that the statistical quantities of the spectrum $P(s)$ and $\overline{\Delta}_3(L)$ have the form expected for an integrable system.

1. Introduction

Although non-relativistic quantum mechanics is a well understood theory, about two decades ago a question arose which is still not completely answered. We know that chaos in classical mechanics is due to nonlinear terms in the equations of motion. The Schrödinger equation is linear, so there should be no quantum chaos. However, classical mechanics is supposed to be some limit of quantum mechanics, so what is the equivalent of chaos in quantum mechanics? By now quite some theory has been developed to answer that question [1]. The presence of chaos can be seen in the spectrum of the Hamiltonian and its statistical properties. On varying a parameter ϵ of the system, two levels could approach one another. In an integrable system, they will continue to approach and cross when ϵ is changed further, but in non-integrable systems, the levels will avoid crossing: they repel. Much research is being done on this topic of 'quantum chaos' [2]. The assumptions underlying these (and other) predictions are not linked rigorously to the integrable and non-integrable nature, although in most cases they seem to hold. Usually, one investigates chaotic systems and determines the statistical properties of the spectrum. Seldom is an integrable system considered, even though such systems are not as trivial as one might expect.

In this paper we look at the elliptic quantum billiard. This billiard is often taken as a reference system for some non-integrable variants [3, 4], and its integrability is taken for granted. An extensive semiclassical survey, as well as numerical solutions to the exact eigenvalue problem, can be found in [5]. We take a closer look at the subtleties of the

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integrability of this billiard. The existence of the second conserved quantity will be investigated in a limiting scheme, involving a larger class of separable systems. Level crossing will be investigated and two statistical properties of the spectrum, namely the distribution of level spacings $P(s)$ and the rigidity $\overline{\Delta}(L)$ [6, 7] are used to establish whether the system is integrable.

2. The elliptic billiard

The elliptic billiard is defined as a particle moving in a two-dimensional potential well with an elliptic boundary. Classically, this system has a second constant of motion: the product of the angular momentum l_1 with respect to one focal point and the angular momentum l_2 with respect to the other focal point [3–5, 8]. This quantity has the same value before and after a collision of the particle with the wall, as well, of course, as during its rectilinear motion. This means that the system is integrable, but there are some subtle points that have not been noticed in the literature.

In the quantum version, the Hamiltonian is that of a free particle on the interior of the ellipse. The wavefunction should vanish on the elliptic boundary that acts as an impenetrable wall. In realistic physical situations, the wall may not be totally impenetrable, which we could mimic by a smooth potential which is very steep around the ellipse. The boundary condition is replaced by the normalization condition. As we make the potential steeper, we expect the system to look more like the billiard. We formulate the smooth problem in elliptic coordinates:

$$x = f \cosh z \cos\theta$$
$$y = f \sinh z \sin\theta$$

so that the focal points are at $(-f, 0)$ and $(f, 0)$. Lines of constant θ are hyperbolae. Lines of constant z are ellipses. The boundary is the ellipse $z = z_b$, of which the eccentricity is $\epsilon = 1/\cosh z_b$. The limit to circular coordinates can be obtained by putting $r = \frac{1}{2} f \exp(z)$, and letting f tend to zero while r remains finite. Defining

$$M(z, \theta) = \cosh^2 z - \cos^2\theta$$

the Hamiltonian $\mathcal{H}$ and $\mathcal{L} \equiv (l_1\, l_2 + l_2\, l_1)/2$ take the form

$$\mathcal{H} = \frac{1}{2mf^2 M(z,\theta)}(p_z^2 + p_\theta^2) + V(z,\theta)$$

and

$$\mathcal{L} = \frac{1}{M(z,\theta)}(\sinh^2 z\ p_\theta^2 - \sin^2\theta\ p_z^2).$$

where $p_z = -\mathrm{i}\hbar\partial_z$ and $p_\theta = -\mathrm{i}\hbar\partial_\theta$. The smooth potential $V(z,\theta)$ is almost zero for $z < z_b$ and very large for $z > z_b$. A conserved quantity should commute with $\mathcal{H}$, but in fact

$$[\mathcal{H}, \mathcal{L}] = -\frac{\hbar \sin^2\theta}{M(z,\theta)}(\hbar\partial_z^2 V + 2\mathrm{i} p_z \partial_z V) + \frac{\hbar \sinh^2 z}{M(z,\theta)}(\hbar\partial_\theta^2 V + 2\mathrm{i} p_\theta \partial_\theta V).$$

so in general $\mathcal{L}$ is not conserved. As the potential becomes steeper, $\partial_z V \to \infty$, so we cannot even see if $\mathcal{L}$ becomes conserved in this limit, and one might question the integrability of the system. For the circular billiard, this problem does not arise: for any potential which only depends on r, the angular momentum commutes with the Hamiltonian. Ayant and Arvieu [9] calculated a few of the lowest-energy eigenvalues of the elliptic billiard and plotted them as a function of the eccentricity. Repelling levels are seen—a sign of non-integrability. Traiber *et al* [10] have shown numerically that these repulsions are actually crossings. They admit, however, that the crossings they find have not been established rigorously. In section 3 we will show how the picture of the integrable billiard as a limit of steep but smooth potentials can be restored.

The billiard problem, given by $V = 0$ and $\Psi(z = z_b) = 0$, is separable in elliptic coordinates. If we substitute

$$\Psi(z, \theta) = N(z)\Theta(\theta)$$

$$E = \frac{2\hbar^2 q}{mf^2}$$

in the time-independent Schrödinger equation $\mathcal{H}\Psi = E\Psi$, we obtain

$$\partial_\theta^2 \Theta + (a - 2q \cos 2\theta)\ \Theta = 0 \tag{1}$$

$$\partial_z^2 N - (a - 2q \cosh 2z)\ N = 0 \tag{2}$$

in which a is a separation constant. Because a and q appear in both equations the eigenvalue problem is not easily soluble (it also raises computational problems [5, 10]). These equations are called the Mathieu equation and the modified Mathieu equation, respectively. Their solutions are Mathieu functions [11, 12]. Due to symmetry, we can restrict ourselves to one quadrant, imposing Dirichlet or Neumann boundary conditions on the x-axis and the y-axis. This gives the standard four classes of solutions. The condition for Θ at $\theta = 0$ and for N at $z = 0$ are both the same as the boundary condition on the x-axis. The condition for Θ at $\theta = \pi/2$ is the boundary condition on the y-axis. Furthermore, N should satisfy the Dirichlet condition at $z = z_b$. If we fix q, there exist countably many values of a for which equation (1) has a solution. Solutions satisfying Neumann (Dirichlet) conditions at $\theta = 0$ are called ce_m (se_{m+1}). The index m runs from zero to infinity. If m is even, the solution satisfies the Neumann condition at $\theta = \pi/2$. If it is odd, the Dirichlet condition is satisfied.

3. Separability

We return to the smooth problem and make an ansatz for a conserved quantity Z in the classical system of the form $Z = \mathcal{L} + 2mf^2\ Y(z, \theta)$. We require that

$$\dot{Z} = \frac{p_z(\partial_z Y + \partial_z V \sin^2\theta) + p_\theta(\partial_\theta Y - \partial_\theta V \sinh^2 z)}{M(z, \theta)/2}$$

be zero for all (p_z, p_θ). From $\partial_z\partial_\theta Y = \partial_\theta\partial_z Y$ we find that V has to be of the special form

$$V(z, \theta) = \frac{V_1(z) + V_2(\theta)}{M(z, \theta)}.$$

This is the class of separable systems [13, 14] of which the elliptic billiard is a limiting case. Y is given by

$$Y(z, \theta) = \frac{V_2(\theta)\ \sinh^2 z - V_1(z)\ \sin^2\theta}{M(z, \theta)}.$$

It can be shown that $[\mathcal{H}, Z] = 0$ for all smooth choices of V_1 and V_2. In the limit of the elliptic billiard, $V_2 \equiv 0$ and V_1 is taken to be zero inside the ellipse and infinite outside. Then Y is formally equal to V. If in the limit the infinite potential V is to be replaced by Dirichlet boundary conditions on eigenfunctions of $\mathcal{H}$, the similar Y contribution to Z is to be replaced by Dirichlet boundary conditions on the eigenfunctions of $\mathcal{L}$. The eigenfunctions of Z will lie in the same Hilbert space as those of $\mathcal{H}$. In fact, the eigenvalue problem of $\mathcal{L}$ is equivalent to that of $\mathcal{H}$: we end up with the same equations (1) and (2). The eigenvalues of $\mathcal{L}$ are given by $(a - 2q)\hbar^2$. Therefore all solutions of these equations are eigenfunctions of both $\mathcal{H}$ and $\mathcal{L}$. Thus they form a basis on which both operators are diagonal, so the two operators commute. This equivalence between the eigenvalue problems, however, also means that $\mathcal{L}$ is of no help in finding the general solution.

There are only four types of billiards in two dimensions that have a second constant of motion which is quadratic in the momenta [13] and have non-complex Hamiltonians. They correspond to rectangles, circles, ellipses and hyperbolae, and parabolae. The parabolic billiard, which has a boundary composed of two opposite parabolae with the same focal point, also has the subtleties of coupled separated equations like equations (1) and (2) and a commutator of a classically conserved quantity with a smooth Hamiltonian, which is zero only in a specific limiting procedure.

4. Characteristic curves

It is possible to use the separability of the system to explain why crossings occur. For that we need to view equation (1) as an eigenvalue problem, with a the eigenvalue and q some parameter. This boundary value problem is of the Sturm–Liouville type, so the spectrum contains an infinite, countable number of only simple eigenvalues bounded from below [15]. We denote these eigenvalues by $a_m(q)$, where m is the same index as in section 2 and q indicates the dependence of the eigenvalue on the parameter q. From the simplicity of the eigenvalues it follows that they depend at least piecewise continuously on q. Overall continuity can be deduced by performing a small rotation $(a', q') = R_\phi(a, q)$ in equation (1) with R_ϕ a rotation over an arbitrary but sufficiently small angle ϕ. This again gives a Sturm–Liouville problem, so in the rotated frame, $a'_m(q')$ has to be piecewise continuous too, and $a_m(q)$ cannot be discontinuous. Equation (2) can also be seen as an eigenvalue problem of the Sturm–Liouville type, but with q as the eigenvalue and a as the parameter. We denote the eigenvalues of this problem with $q_r(a)$, where the index r runs from one to infinity. The $q_r(a)$ can also be seen to be continuous.

We can consider the graphs of the eigenvalues $a_m(q)$ as a set of lines in the (q, a)-plane which do not intersect, and we call those the a-curves. The same picture can be used for the graphs of $q_r(a)$, which are the q-curves. Since the values of q and a in the two equations have to agree, a solution to the problem exists for every intersection point of the two sets of curves. The values of m and r can be considered the quantum numbers of that solution. We determined some of the lower ones of these so-called characteristic curves numerically, using a discretization of equations (1) and (2) and applying the QL algorithm on the resulting tri-diagonal matrices [16]. For equation (2) we took the boundary at $z_b = 2$, corresponding to an eccentricity ϵ of $1/(\cosh 2)$. The results are plotted in figure 1. The eigenvalue of the Hamiltonian is proportional to the q-value, i.e. the projection of the intersections of the a- and q-curves on the q-axis. If two points are close together in projection on the q-axis, this does not mean that they are close in the (q, a)-plane. When ϵ is changed, the q-curves shift and the intersection points move. The projections of two points can move towards each other, but that does not in general correspond to approaching points or any other special case in the (q, a)-plane, so they will continue to move in the same direction when ϵ is changed further. Thus they will cross.

We can now understand the different results of Ayant and Arvieu [9] and Traiber *et al* [10]. Traiber *et al* [10] used an algorithm which enabled them to calculate the a value for given q numerically, which are in effect the a-curves. Via a kind of Newton–Raphson procedure they found the eigenvalues q of the modified Mathieu equation. From the above discussion, it is no surprise that in their figure the levels cross. Ayant and Arvieu [9] did not obtain the eigenvalues one by one. They chose a basis of the Hilbert space to turn the eigenvalue problem for $\mathcal{H}$ into that of a matrix. Truncation of this matrix gives a finite one, of which the eigenvalues can be calculated numerically. Due to roundoff errors, a diagonalization routine can gives spurious repulsions. Ayant and Arvieu [9] do not say what kind of diagonalization method they used. As is shown in figure 2, using a method that can handle degeneracies (first applying the Householder method to obtain a tri-diagonal matrix, then applying the QL algorithm [16]), one finds the correct crossings that were also found by Traiber *et al* [10] in a different way. The matrix size was 98×98 and $\mu = 1/\sqrt{1-\epsilon^2}$.

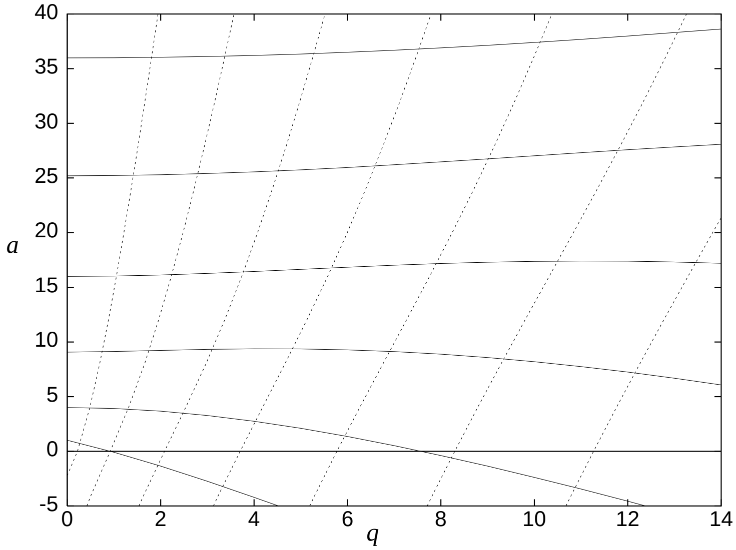

Figure 1. The two independent sets of characteristic curves. The solid curves are the a-curves corresponding to the solutions se_{m+1}, and the dashed curves are the q-curves for eccentricity $\epsilon = 1/(\cosh 2)$.

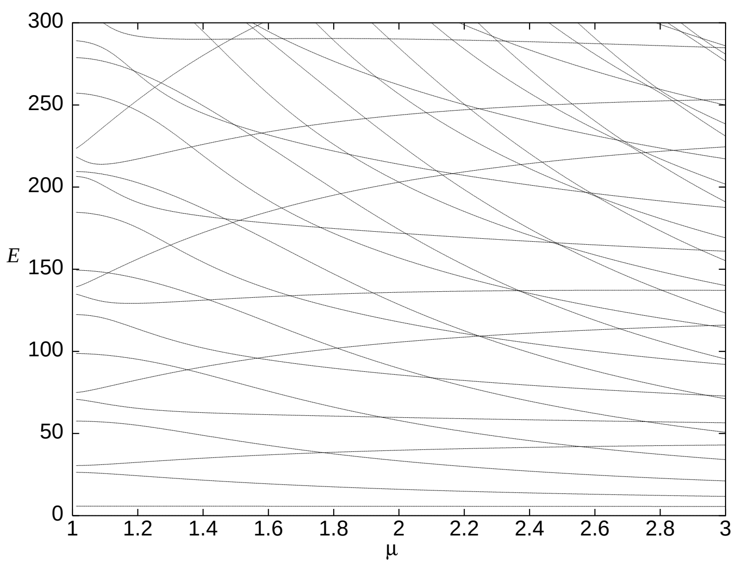

Figure 2. Crossing lower-energy levels as a function of $\mu = 1/\sqrt{1-\epsilon^2}$. The energy is given in units of $(\hbar^2/2mf^2)(\mu - \mu^{-1})$, as in Ayant and Arvieu [9] and Traiber *et al* [10].

5. Asymptotic results

According to current theory [6], random matrix theory can be used for non-integrable systems. One finds that $P(s) = \frac{1}{2}\pi \;\; s \;\; \mathrm{e}^{-\frac{1}{4}\pi s^2}$ and that $\overline{\Delta}(L)$ grows logarithmically with L in the 'Gaussian orthogonal ensemble'. The fact that $P(0) = 0$ is a sign of level repulsion. For integrable systems one expects that $P(s) = \mathrm{e}^{-s}$, which is the distribution of level spacings in the case where the levels are Poissonian distributed, and that $\overline{\Delta}(L)$ grows as $L/15$, for non-degenerate levels, up to a saturation point beyond which $\overline{\Delta}(L)$ remains constant [7]. A reliable calculation of $P(s)$ and $\overline{\Delta}(L)$ requires many energy levels. We will use an asymptotic approach to calculate the high-energy eigenvalues. We follow the Horn-Jeffreys method as in McLachlan [11] and Arscott [12]. We write $a(q)$ as an asymptotic expansion in powers of $k = \sqrt{q}$:

$$a = -2k^2 + 2(2m+1)k + \alpha_0 + \sum_{i=1}^{\infty} \alpha_i k^{-i}.$$

The asymptotic form of the Mathieu equation can be written as the equation for the harmonic oscillator, hence the integer constant m. This m is the same index as before. This a is asymptotically on the a_m-curves corresponding to the solutions ce_m and se_{m+1}. For the expansion of Θ we use

$$\Theta(\theta) \sim \mathrm{e}^{k\chi(\theta)}\zeta(\theta)\left[1 + \sum_{i=1}^{\infty} k^{-i} f_i(\theta)\right].$$

These expressions are substituted in equation (1) and terms of equal powers in k are equated. There are two independent solutions. The first is given by

$$\begin{aligned} \zeta(\theta) &= [\cos\theta\ \tan^{2m+1}(\theta/2 + \pi/4)]^{-1/2} \\ \chi(\theta) &= 2\sin\theta \\ f_{i+1}(\theta) &= -\int^{\theta} \frac{\partial^2_{\theta'}(f_i\zeta) + \zeta\ \sum_{j=0}^{i} \alpha_j f_{i-j}}{4\zeta\cos\theta'}\,\mathrm{d}\theta' \end{aligned} \tag{3}$$

where, by definition, $f_0 \equiv 1$. In [11] only terms up to f_0 are included to find eigenvalues. The spectrum that is found is equivalent to a two-dimensional harmonic oscillator. Berry and Tabor [17] have calculated $P(s)$ for this system. For some ratios of the frequencies, $P(s)$ is not defined. For other ratios, $P(s)$ shows some peeked behaviour, but not a e^{-s} behaviour. They also showed that $P(s)$ can again approach e^{-s} when the system is perturbed. Including f_1 could have the same effect. From equation (3) we find

$$f_1(\theta) = \frac{1}{8}\left[\frac{-(m^2+m+1)\sin\theta + 2m + 1}{\cos^2\theta} - \left(m^2 + m + \frac{1}{2} + 2\alpha_0\right)\log\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right].$$

In order to obtain periodic solution we have to set the logarithmic term equal to zero, so $\alpha_0 = -(2m^2 + 2m + 1)/4$. This is the general strategy for obtaining the α_i's. By induction from equation (3) the general form of f_i can be seen to be

$$f_i(\theta) = \sum_{j=1}^{i} \frac{b_j^{(i)} + a_j^{(i)}\sin\theta}{\cos^{2j}\theta}.$$

The second independent solution of equation (1) is found by substituting $-\theta$ for θ. For ce-type solutions, the boundary condition at $\theta = 0$ can be fulfilled using $ce_m \propto \Theta(\theta) + \Theta(-\theta)$. The modified Mathieu equation (2) can be found from the standard Mathieu equation (1) by substitution of $\mathrm{i}z$ for θ. The resulting solution is called $Ce_m(z)$. Thus $Ce_m(z) \propto \Theta(\mathrm{i}z) + \Theta(-\mathrm{i}z)$ is a solutions satisfying the condition at $z = 0$. The eigenvalues are now given by the Dirichlet boundary condition at $z = z_b$, so that the phase $\Phi(z_b)$ of $\Theta(\mathrm{i}z_b)$ should be $(r + \gamma)\pi$, where r is the same index as in section 4 and $\gamma = \frac{1}{2}$. For se-type solutions, we start with $se_m \propto \Theta(\theta) - \Theta(-\theta)$, and we find the same requirement, but with $\gamma = 0$. The phase can be expressed in terms of ϵ and the $a_j^{(i)}$ and $b_j^{(i)}$:

$$\Phi(z_b) \sim 2k\frac{\sqrt{1-\epsilon^2}}{\epsilon} - (2m+1)\arctan\sqrt{\frac{1-\epsilon}{1+\epsilon}} + \arctan\left[\frac{\sqrt{1-\epsilon^2}}{\epsilon}\frac{\sum_i\sum_j a_j^{(i)}\epsilon^{2j}k^{-i}}{1 + \sum_i\sum_j b_j^{(i)}\epsilon^{2j}k^{-i}}\right]$$

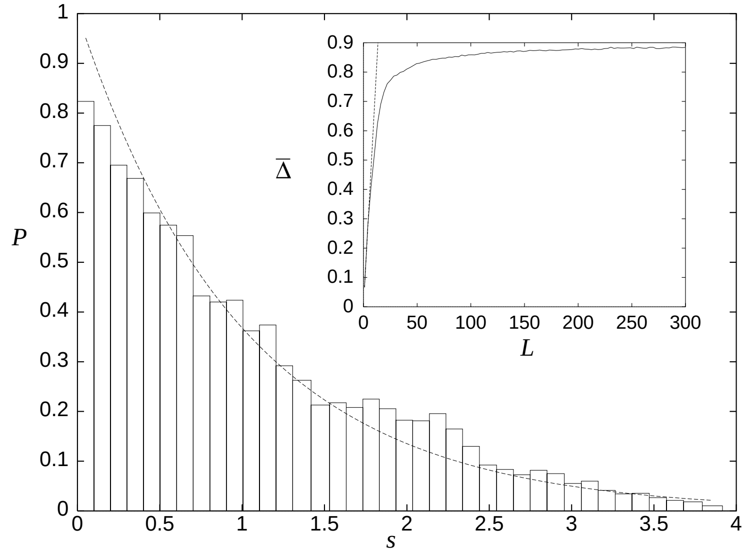

Figure 3. $P(s)$ for eccentricity 0.80. The bars are the calculated points; the dashed line is the theoretical prediction for an integrable system. The inset shows $\overline{\Delta}(L)$ for the same eccentricity. The solid line consists of calculated points; the dashed line is the theoretical prediction $\overline{\Delta}(L) = L/15$ for small L for integrable (non-degenerate) systems. For large L the prediction is that $\overline{\Delta}(L)$ saturates.

which should be equal to $(r+\gamma)\pi$. Using the form of f_1, we obtain the first-order equation for k:

$$k = (r+\gamma)\,\omega_1 + \left(m+\frac{1}{2}\right)\frac{\omega_2}{2} + \frac{\omega_1}{\pi}\arctan\left[\epsilon\sqrt{1-\epsilon^2}\frac{m^2+m+1}{8k+\epsilon^2(2m+1)}\right] \tag{4}$$

where

$$\omega_1 = \frac{\pi\epsilon}{2\sqrt{1-\epsilon^2}} \qquad \frac{\omega_2}{\omega_1} = \frac{4}{\pi}\arctan\sqrt{\frac{1-\epsilon}{1+\epsilon}}.$$

The accuracy improves as k becomes larger and ϵ gets closer to one. For $\epsilon = 0$, corresponding to the circle, it is not a good approximation. Equation (4) is a transcendental equation for k, to be solved for each pair of quantum numbers m and r. The lowest-order eigenvalues, given by the first two terms in (4), form a set of lines in the (ϵ, k)-plane, one line for every pair (r, m). Lines with equal m but different r are shifted in the k direction by a multiple of ω_1, which is not zero except at $\epsilon = 0$, so they will never cross for $\epsilon > 0$. However, lines with different m do cross, at least to lowest order. The correction term in equation (4) can be seen to be at most $\omega_1/2$. This determines a band in the (k, ϵ)-plane to which the lines are certainly confined. If the lines remain continuous when all orders are taken into account, then they have to intersect in some point in the area where these bands overlap. If k is determined by $f(k, \epsilon) = 0$, the implicit function theorem states that $k(\epsilon)$ is continuous provided that $\partial_k f(k, \epsilon) \neq 0$. One can easily check that this is the case for equation (4), so the solution is continuous and crossing is inevitable.

We solved equation (4) numerically, for about 15 000 levels of the *ce*-type, for even m. We took the 10 000 largest of those to compute $P(s)$ and $\overline{\Delta}(L)$. For the unfolding of the spectrum [18] we took for the accumulated level density

$$N(k) = \frac{(k+\omega_2/2-\omega_1)^2 - (\omega_1^2+\omega_2^2)/12}{2\omega_1\omega_2}$$

which follows from the eigenvalues calculated to lowest order. The results are shown in figure 3 for eccentricity $\epsilon = 0.8$. We see the expected behaviour for integrable systems. The graphs look roughly alike for all other values of ϵ, although for some values of the eccentricity, the first correction term in equation (4) cannot totally restore the e^{-s} behaviour, namely when ω_2/ω_1 is a rational number $z = p/q$, which is at $\epsilon = \cos(z\pi/2)$. This is most pronounced for ratios z of $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{2}{3}$.

6. Conclusions

It is possible to define a second constant of motion for the elliptic billiard, as a limiting case using smooth potentials, which are included in this quantity. Separability does not mean we can solve the system, but it does provide a geometric picture in which the energy eigenvalues are projections of intersections of characteristic curves. As the curves change continuously when the eccentricity is varied, the energy levels will cross generically. The level repulsions found in Ayant and Arvieu [9] were not correct, due to the diagonalization method used. Traiber *et al* [10] effectively used the characteristic curves, and therefore the crossing levels that we expect were found. The separability also allows for an asymptotic method to obtain the spectrum, which does indeed give results characteristic of integrable systems. So the elliptic billiard turns out to be an ordinary integrable system, despite the subtleties in the formalism.

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